Technical Report:

Optimizing Dirty Paper Coding for Multiuser MIMO Systems with Rank Constraints

Duy H. N. Nguyen and R. W. Heath, Jr.

I. Introduction

The capacity achieving coding scheme for downlink transmission in a multiuser MIMO system is the dirty-paper coding (DPC) [1], [2]. We consider a system with K users, each equipped with N_{r} antennas being served by a N_{t}-antenna base-station (BS). With DPC, the capacity limit of the multiuser broadcast channel (BC) is found through the optimization [3], [4]:

$$\begin{aligned} \underset{\mathbf{Q}_1,\ldots,\mathbf{Q}_K}{\text{maximize}} \quad & \sum_{i=1}^{K} \log \frac{\left|\mathbf{I}_{N_{\mathrm{r}}} + \mathbf{H}_i \left(\sum_{j=1}^{i} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|}{\left|\mathbf{I}_{N_{\mathrm{r}}} + \mathbf{H}_i \left(\sum_{j=1}^{i-1} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|} \\ \text{subject to} \quad & \sum_{i=1}^{K} \mathrm{Tr}\left\{\mathbf{Q}_i\right\} \leq P \end{aligned} \tag{1}$$

where $\mathbf{H}_i$ is the downlink channel to user-i, $\mathbf{Q}_i$ is the transmit covariance matrix for user-i, $\mathbf{I}$ is the Gaussian noise covariance matrix and P is power constraint at the BS. Herein, we assume an encoding order from user-K to user-1 so that a codeword intended for user-i does not see the interference from user-K to user-$(i+1)$. Note that the formulation in (1) requires abstracting the receiver operation and the number of data streams to each users. In contrast, with a linear coding scheme and the minimization of weighted mean squared error (WMMSE) algorithm [5], the number of data streams for each user can be specified through the dimension of the precoders. To provide a fair comparison between different fully digital, two-stage sparse hybrid, and proposed hybrid (W)MMSE precoding/combining schemes, a same limit on the number of data streams must be set in deriving their algorithmic solutions. In this case, the number of data streams for user-i is $N_{\mathrm{r}}^{\mathrm{RF}}$.

Although problem (1) is nonconvex. it can be optimally solved [4], [6]. At its optimal solution, the rank of the covariance matrix $\mathbf{Q}_i$ then determines the number of data streams transmitted to user-i. If the number of data streams for user-i is capped at $N_{\mathrm{r}}^{\mathrm{RF}}$, problem (1) becomes

$$\begin{aligned} \underset{\mathbf{Q}_1,\ldots,\mathbf{Q}_K}{\text{maximize}} \quad & \sum_{i=1}^{K} \log \frac{\left|\mathbf{I}_{N_{\mathrm{r}}} + \mathbf{H}_i \left(\sum_{j=1}^{i} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|}{\left|\mathbf{I}_{N_{\mathrm{r}}} + \mathbf{H}_i \left(\sum_{j=1}^{i-1} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|} \\ \text{subject to} \quad & \sum_{i=1}^{K} \mathrm{Tr}\left\{\mathbf{Q}_i\right\} \leq P \\ & \mathrm{rank}\left\{\mathbf{Q}_i\right\} \leq N_{\mathrm{r}}^{\mathrm{RF}}. \end{aligned} \tag{2}$$

The nonconvex rank constraints then make the problem even harder to solve than the original problem (1). For a single user system, an optimal solution to problem (2) can be found by waterfilling to the $N_{\mathrm{r}}^{\mathrm{RF}}$ strongest eigenmodes. However, no work has tackled problem (2) in a multiuser setting. To this end, we attempt to provide an algorithmic solution to problem (2) via two main steps: 1) proving the equivalence between problem (2) and an optimization in the multiple-access channel (MAC) using the uplink-downlink duality, and 2) devising numerical algorithms to solve the MAC problem.

II. The Equivalent MAC Problem

Lemma 1. *Given a covariance matrix $\mathbf{\Sigma}$ for some channel $\mathbf{H}$, there exists a covariance matrix $\overline{\mathbf{\Sigma}}$ such that* $\mathrm{tr}\{\overline{\mathbf{\Sigma}}\} \leq \mathrm{tr}\{\mathbf{\Sigma}\}$, $\mathrm{rank}\{\overline{\mathbf{\Sigma}}\} \leq \mathrm{rank}\{\mathbf{\Sigma}\}$ *and*

$$\log\left|\mathbf{I} + \mathbf{H}\mathbf{\Sigma}\mathbf{H}^*\right| = \log\left|\mathbf{I} + \mathbf{H}^*\overline{\mathbf{\Sigma}}\mathbf{H}\right|. \tag{3}$$

Proof: This lemma arises from the concept of *flipped channel* in [3] with the additional consideration of the matrix rank. If the singular value decomposition of $\mathbf{H}$ is $\mathbf{H} = \mathbf{F}\mathbf{\Lambda}\mathbf{G}^*$, then one can choose $\overline{\mathbf{\Sigma}} = \mathbf{F}\mathbf{G}^*\mathbf{\Sigma}\mathbf{G}\mathbf{F}^*$. Clearly, $\mathrm{rank}\{\mathbf{\Sigma}\} \leq \mathrm{rank}\{\overline{\mathbf{\Sigma}}\}$. The proof of $\log\left|\mathbf{I} + \mathbf{H}\mathbf{\Sigma}\mathbf{H}^*\right| = \log\left|\mathbf{I} + \mathbf{H}^*\overline{\mathbf{\Sigma}}\mathbf{H}\right|$ and $\mathrm{rank}\{\mathbf{\Sigma}\} \leq \mathrm{rank}\{\overline{\mathbf{\Sigma}}\}$ are given in Appendix A of [3]. ∎

Consider a MAC with $\mathbf{H}_i^* = \mathbf{H}_i$ as the uplink channel from user-i to the BS. Denote $\mathbf{P}_i$ as the transmit covariance matrix of user-i in the MAC. We assume successive interference cancellation in the MAC where user-1 is decoded first, followed by user-2 and continuously until user-K.

Let

$$\mathbf{A}_i \triangleq \mathbf{I}_{N_{\mathrm{r}}} + \mathbf{H}_i \left(\sum_{j=1}^{i-1} \mathbf{Q}_j\right) \mathbf{H}_i^* \tag{4}$$

$$\mathbf{B}_i \triangleq \mathbf{I}_{N_{\mathrm{t}}} + \sum_{j=i+1}^{K} \mathbf{H}_j^* \mathbf{P}_j \mathbf{H}_j \tag{5}$$

Denote R_i^{MAC} as the achievable data rate for user-i in the MAC, which is given by

$$R_i^{\mathrm{MAC}} = \log \frac{\left|\mathbf{I}_{N_\mathrm{t}} + \sum_{j=i}^{K} \mathbf{H}_j^* \mathbf{P}_j \mathbf{H}_j\right|}{\left|\mathbf{I}_{N_\mathrm{t}} + \sum_{j>i}^{K} \mathbf{H}_j^* \mathbf{P}_j \mathbf{H}_j\right|} = \log \left|\mathbf{I}_{N_\mathrm{t}} + \mathbf{B}_i^{-1} \mathbf{H}_i^* \mathbf{P}_i \mathbf{H}_i\right|. \tag{6}$$

Likewise, denote R_i^{BC} as the achievable data rate for user-i in the BC, which is given by

$$R_i^{\mathrm{BC}} = \log \frac{\left|\mathbf{I}_{N_\mathrm{r}} + \mathbf{H}_i \left(\sum_{j=1}^{i} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|}{\left|\mathbf{I}_{N_\mathrm{r}} + \mathbf{H}_i \left(\sum_{j=1}^{i-1} \mathbf{Q}_j\right) \mathbf{H}_i^*\right|} = \log \left|\mathbf{I}_{N_\mathrm{r}} + \mathbf{A}_i^{-1} \mathbf{H}_i^* \mathbf{Q}_i \mathbf{H}_i\right| \tag{7}$$

Lemma 2. *Given an arbitrary set of uplink covariance matrix* $\mathbf{P}_1, \ldots, \mathbf{P}_K$*, it is possible to find a set of downlink covariance matrices* $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ *such that* $\sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{Q}_i\} \leq \sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{P}_i\}$*,* $R_i^{\mathrm{BC}} = R_i^{\mathrm{MAC}}$*, and* $\mathrm{rank}\{\mathbf{Q}_i\} \leq \mathrm{rank}\{\mathbf{P}_i\}$.

Proof: This lemma arises from the MAC-BC transformation in [3] with the additional consideration of the matrix rank. In this transformation, if $\mathbf{Q}_i$ is chosen as

$$\mathbf{Q}_i = \mathbf{B}_i^{-1/2} \overline{\mathbf{A}_i^{1/2} \mathbf{P}_i \mathbf{A}_i^{1/2}} \mathbf{B}_i^{-1/2} \tag{8}$$

then $R_i^{\mathrm{BC}} = R_i^{\mathrm{MAC}}$ [3]. Herein, $\overline{\mathbf{A}_i^{1/2} \mathbf{P}_i \mathbf{A}_i^{1/2}}$ is obtained from flipping the channel $\mathbf{B}_i^{-1/2} \mathbf{H}_i^* \mathbf{A}_i^{-1/2}$ with the covariance matrix $\mathbf{A}_i^{1/2} \mathbf{P}_i \mathbf{A}_i^{1/2}$. Then, we have

$$\mathrm{rank}\{\mathbf{Q}_i\} \leq \mathrm{rank}\{\overline{\mathbf{A}_i^{1/2} \mathbf{P}_i \mathbf{A}_i^{1/2}}\} \leq \mathrm{rank}\{\mathbf{A}_i^{1/2} \mathbf{P}_i \mathbf{A}_i^{1/2}\} = \mathrm{rank}\{\mathbf{P}_i\} \tag{9}$$

where the second equality is due to Lemma 1 and the equality is due to the full rank matrix $\mathbf{A}_i$. ■

Lemma 3. *Given an arbitrary set of downlink covariance matrix* $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$*, it is possible to find a set of uplink covariance matrices* $\mathbf{P}_1, \ldots, \mathbf{P}_K$ *such that* $\sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{P}_i\} \leq \sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{Q}_i\}$*,* $R_i^{\mathrm{MAC}} = R_i^{\mathrm{BC}}$*, and* $\mathrm{rank}\{\mathbf{P}_i\} \leq \mathrm{rank}\{\mathbf{Q}_i\}$.

Proof: The proof of this lemma is similar to that of Lemma 2 using the BC-MAC transformation in [3]. ■

Consider the sum-rate maximization in MAC with the rank constraints as follows:

$$\begin{aligned} \underset{\mathbf{P}_1, \ldots, \mathbf{P}_K}{\mathrm{maximize}} \quad & \log \left|\mathbf{I}_{N_\mathrm{t}} + \sum_{i=1}^{K} \mathbf{H}_i^* \mathbf{P}_i \mathbf{H}_i\right| \\ \text{subject to} \quad & \sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{P}_i\} \leq P \\ & \mathrm{rank}\{\mathbf{P}_i\} \leq N_\mathrm{r}^{\mathrm{RF}}. \end{aligned} \tag{10}$$

One direct implication of Lemmas 2 and 3 is that the optimal solution of problem (10) can induce the optimal solution of problem (2) via the MAC-BC transformation. Thus, we are interested in solving problem (10) with the following two approaches.

III. Algorithmic Solutions to the MAC Problem

A. Sum Power Iterative Waterfilling

The sum power iterative waterfilling algorithm [4] can be modified to accommodate the rank constraints as follows:

1) Generate the effective channels

$$\mathbf{G}_i^{(n)} = \mathbf{H}_i \left(\mathbf{I}_{N_\mathrm{t}} + \sum_{j \neq i}^{K} \mathbf{H}_j^* \mathbf{P}_i^{(n-1)} \mathbf{H}_i \right)^{-1/2} \tag{11}$$

   for $i = 1, \ldots, K$.

2) Treating these effective channels as parallel, noninterfering channels, obtain the new covariance matrices $\{\mathbf{P}_i^{(n)}\}_{i=1}^K$ by

$$\{\mathbf{P}_i^{(n)}\}_{i=1}^K = \arg \max_{\substack{\mathbf{P}_i \succeq \mathbf{0}, \sum_{i=1}^K \mathrm{Tr}\{\mathbf{P}_i\} \leq P \\ \mathrm{rank}\{\mathbf{P}_i\} \leq N_\mathrm{r}^\mathrm{RF}}} \sum_{i=1}^{K} \log \left| \mathbf{I}_{N_\mathrm{t}} + \left(\mathbf{G}_i^{(n)}\right)^* \mathbf{P}_i \mathbf{G}_i^{(n)} \right|. \tag{12}$$

   Perform the eigen-decomposition $\mathbf{G}_i^{(n)} \left(\mathbf{G}_i^{(n)}\right)^* = \mathbf{U}_i \mathbf{D}_i \mathbf{U}_i^*$, where $\mathbf{U}_i$ is unitary and $\mathbf{D}_i$ is diagonal. Denote $\tilde{\mathbf{D}}_i$ as an $N_\mathrm{r}^\mathrm{RF} \times N_\mathrm{r}^\mathrm{RF}$ diagonal matrix whose diagonal elements are the N_r^RF strongest eigenvalues in $\mathbf{D}_i$. Then the updated covariance matrices are given by

$$\mathbf{P}_i^{(n)} = \mathbf{U}_i \mathbf{\Lambda}_i \mathbf{U}_i^* \tag{13}$$

   where $\mathbf{\Lambda}_i = \mathrm{blkdiag}\left\{ \left[\mu \mathbf{I}_{N_\mathrm{r}} - \tilde{\mathbf{D}}_i^{-1} \right]^+, \mathbf{0}_{N_\mathrm{r} - N_\mathrm{r}^\mathrm{RF}} \right\}$ and the operation $[\mathbf{X}]^+$ denotes a component-wise maximum with zero. Here, the water-filling level μ is chosen such that $\sum_{i=1}^K \mathrm{Tr}\{\mathbf{P}_i^{(n)}\} = P$.

B. Dual Decomposition

The sum-rate maximization problem (10) can also be solved via the dual decomposition [6]. Denote the Lagrangian function

$$\mathcal{L}(\mathbf{P}_1, \ldots, \mathbf{P}_K, \lambda) = \log \left| \mathbf{I}_{N_\mathrm{t}} + \sum_{i=1}^{K} \mathbf{H}_i^* \mathbf{P}_i \mathbf{H}_i \right| - \lambda \left(\sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{P}_i\} - P \right) \tag{14}$$

where $\lambda \geq 0$ is the Lagrangian multiplier associated with the power constraint. Then the optimization (10) can be restated as

$$\underset{\lambda \geq 0}{\text{minimize}} \max_{\mathbf{P}_i \succeq \mathbf{0},\, \text{rank}\{\mathbf{P}_i\} \leq N_{\text{r}}^{\text{RF}}} \mathcal{L}(\mathbf{P}_1, \ldots, \mathbf{P}_K, \lambda). \tag{15}$$

For a given λ, $\mathcal{L}(\mathbf{P}_1, \ldots, \mathbf{P}_K, \lambda)$ can be maximized through the following inner loop iteration until convergence:

1) For user-i, generate the effective noise covariance matrix

$$\mathbf{R}_i = \mathbf{I}_{N_{\text{t}}} + \sum_{j \neq i}^{K} \mathbf{H}_j^* \mathbf{P}_j^{(n-1)} \mathbf{H}_j \tag{16}$$

2) Update $\mathbf{P}_i$ through the following optimization

$$\mathbf{P}_i^{(n)} = \underset{\mathbf{P}_i \succeq \mathbf{0},\, \text{rank}\{\mathbf{P}_i\} \leq N_{\text{r}}^{\text{RF}}}{\text{maximize}} \log \left| \mathbf{I}_{N_{\text{t}}} + \mathbf{R}_i^{-1} \mathbf{H}_i^* \mathbf{P}_i \mathbf{H}_i \right| - \lambda \text{Tr}\{\mathbf{P}_i\}. \tag{17}$$

Perform the eigen-decomposition $\mathbf{H}_i \mathbf{R}_i^{-1} \mathbf{H}_i^* = \mathbf{U}_i \mathbf{D}_i \mathbf{U}_i^*$, where $\mathbf{U}_i$ is unitary and $\mathbf{D}_i$ is diagonal. Denote $\tilde{\mathbf{D}}_i$ as an $N_{\text{r}}^{\text{RF}} \times N_{\text{r}}^{\text{RF}}$ diagonal matrix whose diagonal elements are the N_{r}^{RF} strongest eigenvalues in $\mathbf{D}_i$. Then the updated covariance matrices are given by

$$\mathbf{P}_i^{(n)} = \mathbf{U}_i \mathbf{\Sigma}_i \mathbf{U}_i^* \tag{18}$$

where $\mathbf{\Sigma}_i = \text{blkdiag}\left\{ \left[(1/\lambda) \mathbf{I}_{N_{\text{r}}} - \tilde{\mathbf{D}}_i^{-1} \right]^+, \mathbf{0}_{N_{\text{r}} - N_{\text{r}}^{\text{RF}}} \right\}$.

At the outer loop iteration, λ can be easily updated by the bisection method until $\sum_{i=1}^{K} \text{Tr}\{\mathbf{P}_i\} = P$.

References

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